

A Lower Bound on the Renormalized Nelson Model

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Abstract

We provide explicit lower bounds for the ground-state energy of the renormalized Nelson model in terms of the coupling constant α and the number of particles N , uniform in the meson mass and valid even in the massless case. In particular, for any number of particles N and large enough α we provide a bound of the form $-C\alpha^2 N^3 \log^2(\alpha N)$, where C is an explicit positive numerical constant; and if α is sufficiently small, we give one of the form $-C\alpha^2 N^3 \log^2 N$ for $N \geq 2$, and $-C\alpha^2$ for $N = 1$. Whereas it is known that the renormalized Hamiltonian of the Nelson model is bounded below (as realized by E. Nelson) and implicit lower bounds have been given elsewhere (as in a recent work by Gubinelli, Hiroshima, and Lőrinczi), ours seem to be the first fully explicit lower bounds with a reasonable dependence on α and N . We emphasize that the logarithmic term in the bounds above is probably an artifact in our calculations, since one would expect that the ground-state energy should behave as $-C\alpha^2 N^3$ for large N or α , as in the polaron model of H. Fröhlich.

1 Introduction

The Nelson Model describes the interaction of a collection of non-relativistic nucleons with a meson field. The model is ascribed to Edward Nelson, who presented it in some detail, using rudimentary tools from Stochastic Calculus (such as the Itô Formula), for the first time in a conference that attempted to improve the communication between mathematicians and physicists at the time of June, 1963, held in Dedham, Massachusetts [16]. Shortly afterward, Nelson submitted an article to the Journal of Mathematical Physics, which got it published in 1964, where he studied the model in a substantially more organized and systematic way, this time exchanging the stochastic-analytic methods utilized previously for operator-theoretic ones [15]. The Hamiltonian describing this interaction between nucleons and mesons is given by

$$H_{\alpha,\mu}^{N,\Lambda} = - \sum_{n=1}^N \frac{\Delta_n}{2} + \int_{\mathbb{R}^3} \chi_{\Lambda}(k) \omega(k) a_k^{\dagger} a_k dk + \sqrt{\alpha} \sum_{n=1}^N \int_{\mathbb{R}^3} \frac{\chi_{\Lambda}(k)}{\sqrt{\omega(k)}} (e^{ikx_n} a_k + e^{-ikx_n} a_k^{\dagger}) dk. \quad (1.1)$$

A couple of words qualifying the use of symbols above are in order. N is the number of nucleons; the integrals, the laplacians, and the particle-variables are all three-dimensional; Λ , α , and μ are all non-negative real constants; $\omega(k) \equiv \sqrt{k^2 + \mu^2}$; and χ_{Λ} is the indicator function of the ball of radius Λ centered at the origin. H acts on $L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}$, where \mathcal{F} is the Fock space over $L^2(\mathbb{R}^3)$, and a_k, a_k^{\dagger} are the standard annihilation and creation operators for the Fock space, satisfying $[a_k, a_{k'}^{\dagger}] = \delta(k - k')$, and $[a_k, a_{k'}] = [a_k^{\dagger}, a_{k'}^{\dagger}] = 0$.

In terms of spectral properties of H , it is known that for finite Λ , H is self-adjoint and bounded-below, but as Λ goes to ∞ the infimum of the spectrum goes to $-\infty$. It is imperative then to keep Λ finite to make sense of H . Despite this constraint, a stabilizing term can be added to the Hamiltonian, which allows one to take Λ to ∞ while still retaining in the limit a bounded-below operator. This was first discovered by Nelson in the papers cited above. The precise term is

$$Q_{\alpha,\mu}^{N,\Lambda} \equiv \alpha N \int \frac{\chi_{\Lambda}(k) dk}{\omega(k) [k^2/2 + \omega(k)]}, \quad (1.2)$$

and the result is that for all real t , $\exp[it(H_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda})] \rightarrow \exp(it\hat{H}_{\alpha,\mu}^N)$ strongly as $\Lambda \rightarrow \infty$, for some self-adjoint, bounded-below operator \hat{H} . \hat{H} is then the renormalized Hamiltonian of the Nelson model. The term Q comes about after performing a Gross transformation on \hat{H} [15]. The main purpose of this article is to give a numerical lower bound to \hat{H} with an explicit dependence on all α , μ and N . Even though there are implicit lower bounds on \hat{H} elsewhere (see, for example, [9, Corollary 2.18]), ours seems to be the first explicit one; moreover, the methods utilized are novel, as explained below, which makes the computations involved and the final result noteworthy.

Even though a whole family of lower bounds for the renormalized Nelson model Hamiltonian is provided at the end, the general result is not very illuminating, and it is perhaps better at this point to simply state particular bounds that may be derived from it. For example, we will show below that for sufficiently large α and all μ , Λ , and N ,

$$E_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda} \geq -C\alpha^2 N^3 \log^2(\alpha N), \quad (1.3)$$

where $E_{\alpha,\mu}^{N,\Lambda}$ is the ground-state energy of $H_{\alpha,\mu}^{N,\Lambda}$ and C is an explicit positive number; and for small enough α we will find a lower bound of the form $E_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda} \geq -C\alpha^2 N^3 \log^2 N$ when $N \geq 2$, and $E_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda} \geq -C\alpha^2$ when $N = 1$. The ground-state energy is then obtained on the left side of the inequalities by taking $\Lambda \rightarrow \infty$ (see, for example, the discussion preceding [8, Equation (1.2)]). We note that C does not depend on μ , and that the result is valid for all meson masses, including the massless case $\mu = 0$; this, in spite that there is no ground state for the massless Hamiltonian [14]. There is reason to believe that the logarithmic factor above is just an artifact due to, for instance, the similarity of this model with the Fröhlich polaron model [7], whose ground-state energy is known to behave as $-D\alpha^2 N^3$ for large α or N (see [1] and references therein, and also the next two sections in this article). In case the reader worries that no upper bound is provided here, we do give one below, albeit rather trivial,

$$E_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda} \leq 0, \quad (1.4)$$

of which we defer an explanation for later.

As an outline of the proof, the main strategy involved is the use of a unified approach to bounding Hamiltonians from below, presented in great detail in a recent article by the author and Lawrence Thomas [3]. The idea is as follows. The ground-state energy of H (with a finite value of Λ) may be expressed by means of a Feynman-Kac formula

$$\inf \text{spec } H = - \lim_{T \rightarrow \infty} T^{-1} \log \{E[\exp(\mathcal{A}_T)]\}, \quad (1.5)$$

for some Brownian functional \mathcal{A}_T (an L^2 functional of 3D Brownian paths on $[0, T]$) that may be computed explicitly in this case, after integrating the meson field variables, which will be explained later below. E denotes expectation with respect to Brownian motion starting at the origin. A single application of the Clark-Ocone formula, well-known from Malliavin Calculus and Mathematical Finance [17, 11, 10], allows one to rewrite the action as

$$\mathcal{A}_T = E(\mathcal{A}_T) + \int_0^T \rho_t dX_t, \quad (1.6)$$

for some unique, adapted, \mathbb{R}^3 -valued, L^2 process ρ , where X is 3D Brownian motion, and the integral is Itô. An easy supermartingale estimate allows one then to obtain the bound

$$E[\exp(\mathcal{A}_T)] \leq \exp[E(\mathcal{A}_T)] E\left[\exp\left(\frac{p^2}{2(p-1)} \int_0^T \rho_t^2 dt\right)\right]^{1-1/p} \quad (1.7)$$

for any $p > 1$ [3]. The main point here is that $\int_0^T \rho_t^2 dt$ will be in many cases a “better behaved” Brownian functional than \mathcal{A}_T , and then a couple of applications of the procedure above should yield at the end a bounded ρ , which will allow the obtention of a lower bound for E , by bounding ρ by its L^∞ norm. Something more efficient and sophisticated is done at the end [3], but the crude method just outlined conveys the spirit of the idea well enough for our purposes here. Coincidentally, $E(\mathcal{A}_T)$ will happen to be, up to sublinear terms, exactly TQ , as defined above in equation (1.2), which will at the end allow one to obtain the desired bound. This is something to take note of, and should not be overlooked: the fact that $E(\mathcal{A}_T) = TQ + o(T)$ is remarkable, since it uncovers the fact that somehow performing a Clark-Ocone expansion at the functional-integral level amounts to doing the Gross transformation that Nelson considered in the context of operator methods.

The structure of the rest of the article is as follows. Section 2 sets out a combinatorial argument that will allow us to improve the lower bounds by a substantial amount. It is first presented in abstract, and then, as a warm-up in Section 3, immediately applied to the N -body polaron model of Herbert Fröhlich, without interelectronic repulsion. This gives an answer consistent with an easy upper bound (as far as the N and large- α behaviors are concerned), $-0.109\alpha^2 N^3$ [1], which follows from an argument of Pekar [18]. Then, in Section 4 a detailed proof of the general lower bound (and in particular of the large- and small- α examples stated above) is provided, using the argument presented in Section 2.

Before delving into the proofs, we would like to mention that this article is based on part of the Ph.D. thesis of the author [2], and we will refer to it from time to time. We would like to thank as well Oliver Matte, Jacob Schach Møller, and Lawrence Thomas for productive discussions on a range of topics. The author also acknowledges partial support from the Danish Council for Independent Research (Grant number DFF-4181-00221).

2 Optimal block design

Suppose one is faced with the following problem. First, consider the set $I_N \equiv \{1, 2, \dots, N\}$, and then construct $J_N \equiv \{(i, j) \in I_N^2 : i \leq j\}$. We say that a subset of J_N is “non-repeating” if there are no repeated indices among its different items. For instance, if $N = 23$, $\{(1, 1), (2, 2), (3, 4)\}$ is non-repeating but $\{(1, 2), (2, 2), (3, 4)\}$ is “repeating,” since $(1, 2)$ and $(2, 2)$ have the index 2 in common. We will call a partition of J_N made out exclusively of non-repeating elements “separated.” So, for example, a separated partition for J_2 is $\{(1, 1), (2, 2)\} \cup \{(1, 2)\}$ but $\{(1, 1), (1, 2)\} \cup \{(2, 2)\}$ is not separated, since the first subset is repeating. (We will use these names constantly in what follows, which is the reason why we have invented them. The reader should rest assured we have tried to keep the nomenclature burden at a minimum.) We now ask, what is the minimal cardinality of a separated partition of J_N ? The maximal number is certainly the cardinality of J_N , that is, $N(N + 1)/2$, which is of order N^2 , and one wonders if one can improve that order. The answer turns out to be yes, and that the minimal number is N .

Let us prove what we have just asserted. One side of the proof involves showing that there are no separated partitions of size smaller than N : if there were a separated partition of J_N with fewer than N elements, then one could query the partition to realize that this is not possible. For instance, one could ask where $(1, 1)$ is, and that would yield a partition element, say G_1 . Then $(1, 2)$ must be in $G_2 \neq G_1$ (for otherwise the partition would be repeating), and $(1, 3)$ must be in $G_3 \neq G_2, G_1$, etc. Eventually one arrives at $(1, a)$, with a the cardinality of the purported smaller partition (less than N), and then $(1, a + 1)$ is nowhere since the partition has already been exhausted at step number a .

The other side entails finding or proving the existence of a separated partition of size N . We will go a little beyond that, and will actually find all separated partitions of size N , thus completely settling the problem. The answer is that all such partitions will be in one-to-one correspondence with the $N \times N$ commutative latin squares. First let us recall what an $N \times N$ commutative latin square is: This is a function $F : I_N^2 \rightarrow I_N$ with the following three properties: for all i, j, k ,

1. $F(i, j) = F(j, i)$ (F is commutative)
2. $F(i, j) \neq F(i, k)$ (no repetitions along rows)
3. $F(i, j) \neq F(k, j)$ (no repetitions along columns)

Obviously, this is the abstract definition of something very concrete, which is recovered by taking the matrix with entry (i, j) given by $F(i, j)$. The attentive reader will notice that 3 follows from 2, because of 1. We have, however, decidedly kept property 3 there to facilitate understanding. Before proceeding, we answer the obvious question: Do $N \times N$ commutative latin squares exist for every N ? Yes, they do; just take the group operation (or the addition table) of any commutative group of cardinality N , such as \mathbb{Z}_N , the integers modulo N .

Now, let F be an $N \times N$ commutative latin square, and consider its restriction to J_N , which we shall call G . Consider the partition $J_N = G^{-1}\{1\} \cup G^{-1}\{2\} \cup \dots \cup G^{-1}\{N\}$. This partition is separated. To see it, first notice that each of the elements of the partition is non-empty, since $j \mapsto F(1, j)$ is injective and therefore surjective; now, consider the set $G^{-1}\{m\}$ for a fixed $1 \leq m \leq N$. If this set were repeating, there would be a pair $(i, j), (k, l)$ of elements in $G^{-1}\{m\}$ such that at least one of the four possibilities occur: $i = k$, $i = l$,

$j = k$, or $j = l$. If $i = k$, then $j = l$, by Property 2. If $i = l$, recall that G comes from a commutative function (Property 1), and so $G(i, j) = F(i, j) = F(k, i) = F(i, k) = m$, again a contradiction by Property 2. Etcetera. We have thus shown that all commutative latin squares form separated partitions.

There is one bit missing, namely proving that every separated partition gives rise to a commutative latin square. In a sense this has already been done, since all the previous steps are reversible; we can still walk through the proof, however. From a separated partition with N elements $J_N = A_1 \cup \dots \cup A_N$ we form the function $G : J_N \rightarrow I_N$ given by $G(i, j) = k$ if and only if $(i, j) \in A_k$; G may be extended to all of I_N^2 as a commutative function F in the obvious way: $F(i, j) \equiv G(j, i)$ for $i > j$. Only Property 2 needs to be checked, and this follows easily, since if $F(i, j) = F(i, k)$, then assuming that $i \leq j$ and $i \leq k$, then both (i, j) and (i, k) belong to A_k , which is impossible; the remaining cases $i > j$ and $i > k$ are handled similarly, using the definition of F as a commutative extension of G . This concludes the proof.

We have called this section “Optimal block design” since, in a sense, that is precisely what we have just achieved. The design of blocks (or partition elements) is a broad area in the field of Combinatorics, and the reader may refer to [4] for more information on this topic. In that same reference latin squares are also treated in great detail.

3 Preliminary calculations: the polaron model

We show the usefulness of the result proven in the previous section in a model that is simpler to handle at the level of functional integrals, namely the polaron model of Herbert Fröhlich. This is a model in complete analogy to that of Nelson, this time describing the interaction of a non-relativistic electron with the optical phonon modes of a polar crystal. As the electron moves inside the crystal, it distorts the atom lattice locally, and this distortion can be represented through quantized waves, called phonons. One may consider N electrons immersed in the crystal, and this particular model has as Hamiltonian,

$$H_\alpha^N = - \sum_{n=1}^N \frac{\Delta_n}{2} + \int_{\mathbb{R}^3} a_k^\dagger a_k dk + \sqrt{\alpha} \sum_{n=1}^N \int_{\mathbb{R}^3} \frac{1}{|k|} (e^{ikx_n} a_k + e^{-ikx_n} a_k^\dagger) dk, \quad (3.1)$$

acting on $L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}$, where \mathcal{F} is the Fock space over $L^2(\mathbb{R}^3)$, as in the Nelson model case, equation (1.1). Note that no ultraviolet cutoff is used for this Hamiltonian. We do not treat the electrons as fermions, and we make the unphysical assumption that the electrons do not repel each other. There is an explicit Feynman-Kac formula for H_α^N , due first to Feynman by directly integrating the quantum field variables [6, 5], and rederived by Nelson by noting that the field variables are driven by an Olstein-Uhlenbeck process when computing matrix elements [16]. See also the Ph.D. thesis of the author for a direct integration of the phonon field using the Trotter product formula [2]. In any case, one obtains the formula

$$\inf \text{spec } H = - \lim_{T \rightarrow \infty} \frac{1}{T} \log \left\{ E \left[\exp \left(\frac{\alpha}{\sqrt{2}} \sum_{m,n} \int_0^T \int_0^t \frac{e^{-(t-s)}}{|X_t^m - X_s^n|} ds dt \right) \right] \right\}. \quad (3.2)$$

The reader should immediately recognize that this has the form prescribed by equation (1.5). Here $X = (X^1, X^2, \dots, X^N)$ is a $3N$ -dimensional Brownian motion. Let us rewrite the functional integral as follows

$$\begin{aligned} & E \left[\exp \left(\frac{\alpha}{\sqrt{2}} \sum_{m,n} \int_0^T \int_0^t \frac{e^{-(t-s)}}{|X_t^m - X_s^n|} ds dt \right) \right] \\ &= E \left[\exp \left(\frac{\alpha}{\sqrt{2}} \sum_{m=1}^N \int_0^T \int_0^t \frac{e^{-(t-s)}}{|X_t^m - X_s^m|} ds dt + \frac{\alpha}{\sqrt{2}} \sum_{m < n} \int_0^T \int_0^t \frac{e^{-|t-s|}}{|X_t^m - X_s^n|} ds dt \right) \right] \\ &= E \left[\prod_{m=1}^N \exp \left(\frac{\alpha}{\sqrt{2}} \int_0^T \int_0^t \frac{e^{-(t-s)}}{|X_t^m - X_s^m|} ds dt \right) \prod_{m < n} \left(\frac{\alpha}{\sqrt{2}} \int_0^T \int_0^t \frac{e^{-|t-s|}}{|X_t^m - X_s^n|} ds dt \right) \right] \\ &\equiv E \left(\prod_{m \leq n} \Omega_{m,n} \right). \end{aligned} \quad (3.3)$$

We pick then any separated partition of J_N with N elements (the one generated by \mathbb{Z}_N , say), write it as $J_N = K_1 \cup K_2 \cup \dots \cup K_N$, where the K_i 's are non-repeating and disjoint, and then we get, by Hölder's inequality for N elements and equal coefficients,

$$E \left(\prod_{m \leq n} \Omega_{m,n} \right) = E \left(\prod_{i=1}^N \prod_{(m,n) \in K_i} \Omega_{m,n} \right) \leq \prod_{i=1}^N E \left(\prod_{(m,n) \in K_i} \Omega_{m,n}^N \right)^{1/N}. \quad (3.4)$$

But, since each K_i is non-repeating, the elements in $\{\Omega_{m,n}^N : (m,n) \in K_i\}$ are independent, which allows one to immediately rewrite the right end of (3.4) as

$$\prod_{i=1}^N \prod_{(m,n) \in K_i} E(\Omega_{m,n}^N)^{1/N} = \prod_{m \leq n} E(\Omega_{m,n}^N)^{1/N}. \quad (3.5)$$

What we have achieved here is a splitting of the left end of (3.4) into a product of simpler, more manageable pieces, while trying to keep the error introduced in doing so at a minimum. One further simplification allows us to write (3.5) as

$$\begin{aligned} \prod_{m \leq n} E(\Omega_{m,n}^N)^{1/N} &= \prod_{m=1}^N E(\Omega_{m,m}^N)^{1/N} \prod_{m < n} E(\Omega_{m,n}^N)^{1/N} = E(\Omega_{1,1}^N) E(\Omega_{1,2}^N)^{(N-1)/2} \\ &= E \left[\exp \left(\frac{N\alpha}{\sqrt{2}} \int_0^T \int_0^t \frac{e^{-(t-s)}}{|Y_t - Y_s|} ds dt \right) \right] E \left[\exp \left(\frac{N\alpha}{\sqrt{2}} \int_0^T \int_0^T \frac{e^{-|t-s|}}{|Y_t - Z_s|} ds dt \right) \right]^{(N-1)/2}, \end{aligned} \quad (3.6)$$

where Y and Z are independent 3D Brownian motions. We have finally reached a point where we can resort to the argument alluded to in the introduction, namely the one involving the Clark-Ocone formula. Here we will merely state the following particular results, which can be derived easily from Theorems 2.2 and 2.3 in the article of the author and Lawrence Thomas referred to above [3],

$$\begin{aligned} E \left[\exp \left(\beta \int_0^T \int_0^t \frac{e^{-(t-s)}}{|Y_t - Y_s|} ds dt \right) \right] &\leq \exp \left[\left(\frac{\beta^2}{2} + \sqrt{2}\beta \right) T \right], \\ E \left[\exp \left(\beta \int_0^T \int_0^t \frac{e^{-(t-s)}}{|Y_t - Z_s|} ds dt \right) \right] &\leq \exp \left[\frac{\beta^2 T}{4} + o(T) \right]. \end{aligned} \quad (3.7)$$

Now, by first rewriting the right end of the bottom equation of (3.6) in order to fit it into the form prescribed by (3.7),

$$\begin{aligned} &E \left[\exp \left(\frac{N\alpha}{\sqrt{2}} \int_0^T \int_0^T \frac{e^{-|t-s|}}{|Y_t - Z_s|} ds dt \right) \right]^{(N-1)/2} \\ &= E \left[\exp \left(\frac{N\alpha}{\sqrt{2}} \int_0^T \int_0^t \frac{e^{-(t-s)}}{|Y_t - Z_s|} ds dt + \frac{N\alpha}{\sqrt{2}} \int_0^T \int_0^t \frac{e^{-(t-s)}}{|Z_t - Y_s|} ds dt \right) \right]^{(N-1)/2} \\ &\leq E \left[\exp \left(\sqrt{2}N\alpha \int_0^T \int_0^t \frac{e^{-(t-s)}}{|Y_t - Z_s|} ds dt \right) \right]^{(N-1)/4} E \left[\exp \left(\sqrt{2}N\alpha \int_0^T \int_0^t \frac{e^{-(t-s)}}{|Z_t - Y_s|} ds dt \right) \right]^{(N-1)/4} \\ &= E \left[\exp \left(\sqrt{2}N\alpha \int_0^T \int_0^t \frac{e^{-(t-s)}}{|Y_t - Z_s|} ds dt \right) \right]^{(N-1)/2}, \end{aligned} \quad (3.8)$$

and then, by the Cauchy-Schwarz inequality, we obtain that (3.6) may be bounded above as

$$\exp \left[\frac{N^2\alpha^2}{4} + N\alpha + \frac{N^2(N-1)\alpha^2}{4} + o(T) \right] = \exp \left[\frac{N^3\alpha^2}{4} + N\alpha + o(T) \right], \quad (3.9)$$

which leads to the conclusion that

$$\inf \text{spec } H \geq -N\alpha - \frac{N^3\alpha^2}{4}. \quad (3.10)$$

This result is in agreement with an upper bound that can be obtained by plugging in a reasonable trial state for large α , namely $-0.109\alpha^2 N^3$ [1]. Coincidentally, it happens to be a most natural extension of the result obtained when the Clark-Ocone formula is used in the one-particle case, $-\alpha - \alpha^2/4$ [3], which is an improvement over a lower bound by Lieb and Yamazaki, $-\alpha - \alpha^2/3$ [13].

4 Proof of the lower bound for the Nelson model

We provide in this section the proof of the lower bound mentioned in the introduction for the Nelson model. After integrating the field variables, one encounters the following expression:

$$\inf \text{spec } H_{\alpha,\mu}^{N,\Lambda} = - \lim_{T \rightarrow \infty} \frac{1}{T} \log \left\{ E \left[\exp \left(\alpha \sum_{m,n} \int_0^T \int_0^t \frac{\chi_\Lambda(k) e^{-\omega(k)(t-s)}}{\omega(k)} e^{-ik(X_t^m - X_s^n)} ds dt dk \right) \right] \right\}. \quad (4.1)$$

(See, for instance, [2]). We then apply the Clark-Ocone formula. The requirement for this to be valid is that the action \mathcal{A}_T , defined here as the argument of the exponential (4.1), be real and in L^2 . Real-valuedness is immediate, as the sine function is odd, and integration over all k 's will make the imaginary part of the k -integral vanish. Square integrability follows from a simple estimate on a single summand of \mathcal{A}_T ,

$$\left| \int_0^T \int_0^t \frac{\chi_\Lambda(k) e^{-\omega(k)(t-s)}}{\omega(k)} e^{-ik(X_t^m - X_s^n)} ds dt dk \right| \leq \int_0^T \int_0^t \frac{\chi_\Lambda(k) e^{-\omega(k)(t-s)}}{\omega(k)} ds dt dk, \quad (4.2)$$

which is independent of the Brownian path chosen. The first part of the proof will be computing $E(\mathcal{A}_T)$. A whole subsection will be devoted to it.

4.1 Computation of $E(\mathcal{A}_T)$

The expectation of \mathcal{A}_T immediately comes about after using the independence of Brownian motions corresponding to different particles and the fact that $E(e^{ikZ}) = e^{-k^2\sigma^2/2}$ for a 3D Gaussian random variable Z with zero mean and variance σ^2 ,

$$\begin{aligned} E(\mathcal{A}_T) &= \alpha N T \int \frac{\chi_\Lambda(k)}{\omega(k) [k^2/2 + \omega(k)]} dk - \alpha N \int \frac{\chi_\Lambda(k)}{\omega(k) [k^2/2 + \omega(k)]^2} \left(1 - e^{-[k^2/2 + \omega(k)]T} \right) dk \\ &\quad + \alpha N(N-1) \int_0^T \int_0^t \int \frac{e^{-k^2 t/2} e^{-k^2 s/2}}{\omega(k)} e^{-\omega(k)(t-s)} \chi_\Lambda(k) dk ds dt. \end{aligned} \quad (4.3)$$

The first term divided by T is, as has been said already, the renormalizing term, which we have called $Q_{\alpha,\mu}^{N,\Lambda}$. The second term is certainly sublinear, and so may be neglected. The third term also turns out to be sublinear. A simple way of showing this is a rather direct application of the Hardy-Littlewood-Sobolev inequality [12]: by removing both the mass of the meson field and the ultraviolet cutoff, which amounts to bounding $\omega(k) \geq |k|$ and $\chi_\Lambda \leq 1$, we can rewrite the third term as

$$\begin{aligned} &\alpha N(N-1) \int_0^T \int_0^t \int \frac{e^{-k^2 t/2} e^{-k^2 s/2}}{|k|} e^{-|k|(t-s)} dk ds dt \\ &= \alpha N(N-1) E \left(\int_0^T \int_0^t \int \frac{e^{-ik(Y_t - Z_s)}}{|k|} e^{-|k|(t-s)} dk ds dt \right). \end{aligned} \quad (4.4)$$

By explicitly calculating the Fourier transform of $e^{-|k|}/|k|$ we get that this is bounded above by

$$4\pi\alpha N(N-1) \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\chi_{[0,T]}(t) p_t(x) \chi_{[0,T]}(s) p_s(y)}{(t-s)^2 + (x-y)^2} dx dy ds dt, \quad (4.5)$$

where p_t is the 3D heat kernel $p_t(x) = (2\pi t)^{-3/2} e^{-x^2/(2t)}$. By the four-dimensional Hardy-Littlewood-Sobolev inequality we get that this is bounded above by $CT^{3/4}$ for some constant C , which proves that the third term of the expectation is sublinear, as claimed. The upshot then is that $E(\mathcal{A}_T) = Q_{\alpha,\mu}^{N,\Lambda} T + o(T)$. We now proceed to compute and estimate the second term in the Clark-Ocone expansion, namely $\int_0^T \rho_t dX_t$.

4.2 Computation of $\int_0^T \rho_t dX_t$

We have stated what the Clark-Ocone formula says in general in equation (1.6), but no recipe or procedure has been given to compute ρ in specific cases. ρ can be calculated explicitly in this situation as $E(D_t \mathcal{A}_T | \mathcal{F}_t)$, where \mathcal{F}_t is the standard filtration of $3N$ -dimensional Brownian motion $\sigma(X_t : 0 \leq t \leq T)$, and D_t is the so-called Malliavin derivative of \mathcal{A}_T . An introduction to the Malliavin derivative here would take us too far afield. The reader is simply referred to the publication of the author and Thomas [3], where a fairly complete introduction to this operator is introduced, discussed, and applied in certain cases of interest. Enough for our purposes here will be to provide the following prescription for computing this derivative for a certain class of Brownian functionals: Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a smooth function with polynomial growth, and g_1, g_2, \dots, g_m a collection of functions $[0, T] \rightarrow \mathbb{R}^n$ in L^2 . Then, if X represents n -dimensional Brownian motion and $W(h)$ is the Itô integral of an L^2 function h from $[0, T]$ to \mathbb{R}^n , $\int_0^T h_t dX_t$,

$$D_t f[W(g_1), W(g_2), \dots, W(g_m)] = \sum_{i=1}^m \frac{\partial f}{\partial x_i} [W(g_1), W(g_2), \dots, W(g_m)] g_i(t). \quad (4.6)$$

(The “stochastic total derivative of f .”) Even though the formula above does not apply immediately to our current case here, it can easily be adapted to it, by approximating the integrals by Riemann sums. (See [3] for more information.) By noting that each one of the summands of \mathcal{A}_T above is actually real (the imaginary part of the k -integral vanishing), one arrives at the end at the formula

$$\begin{aligned} D_u \mathcal{A}_T &= D_u \left(\alpha \sum_{m,n} \int_0^T \int_0^t \frac{\chi_\Lambda(k) e^{-\omega(k)(t-s)}}{\omega(k)} \cos[k(X_t^m - X_s^n)] ds dt dk \right) \\ &= \alpha \sum_{m,n} \int_0^T \int_0^t \frac{\chi_\Lambda(k) e^{-\omega(k)(t-s)}}{\omega(k)} D_u \cos[k(X_t^m - X_s^n)] ds dt dk. \end{aligned} \quad (4.7)$$

Note how $\cos[k(X_t^m - X_s^n)]$ is of the form $f[W(g)]$, as

$$\cos[k(X_t^m - X_s^n)] = \cos \left\{ \int_0^T [k^m 1_{[0,t]}(r) - k^n 1_{[0,s]}(r)] dX_r \right\}, \quad (4.8)$$

where k^l is the embedding of the vector k into the null vector with $3N$ coordinates, with its components written in the $3l-2$, $3l-1$, and $3l$ positions (that is, $k_{3l-i}^l = k_{3-i}$ for all $0 \leq i \leq 2$ and is zero for all the other coordinates). Therefore, by means of the formula (4.6) (with $m=1$),

$$\begin{aligned} & \int_0^T \int_0^t \frac{\chi_\Lambda(k) e^{-\omega(k)(t-s)}}{\omega(k)} D_u \cos[k(X_t^m - X_s^n)] ds dt dk \\ &= - \int_0^T \int_0^t \frac{\chi_\Lambda(k) e^{-\omega(k)(t-s)}}{\omega(k)} \sin[k(X_t^m - X_s^n)] [k^m 1_{[0,t]}(u) - k^n 1_{[0,s]}(u)] ds dt dk \\ &= -i \int_0^T \int_0^t \frac{\chi_\Lambda(k) e^{-\omega(k)(t-s)}}{\omega(k)} e^{-ik(X_t^m - X_s^n)} [k^m 1_{[0,t]}(u) - k^n 1_{[0,s]}(u)] ds dt dk, \end{aligned} \quad (4.9)$$

and this is how we finally arrive at the formula

$$D_u \mathcal{A}_T = -i\alpha \sum_{m,n} \int_0^T \int_0^t \frac{\chi_\Lambda(k) e^{-\omega(k)(t-s)}}{\omega(k)} e^{-ik(X_t^m - X_s^n)} [k^m 1_{[0,t]}(u) - k^n 1_{[0,s]}(u)] ds dt dk. \quad (4.10)$$

The conditional expectation $E(D_u \mathcal{A}_T | \mathcal{F}_u)$ may now be computed directly, by writing e^{ikY_r} as the product $e^{ik(Y_r - Y_u)} e^{ikY_u}$ for a 3D Brownian motion Y and $r \geq u$, using the independence of X_n and X_m for different n and m , and the Markov property of Brownian motion, obtaining

$$\begin{aligned} & \alpha \sum_{m=1}^N \int k^m \frac{\chi_\Lambda(k)}{\omega(k)} \int_u^T \int_0^t [1 + \Theta(s-u)] e^{-\omega(k)(t-s)} e^{-k^2(t-u)/2} e^{-k^2(s-u)_+/2} \\ & \quad \times \sum_{n=1}^N \sin[k(X_{s \wedge u}^n - X_u^m)] ds dt dk, \end{aligned} \quad (4.11)$$

where Θ is Heaviside's theta function (the indicator function of the set of non-negative numbers). Therefore, $E(D_u \mathcal{A}_T | \mathcal{F}_u)^2$ is equal to

$$\begin{aligned} & \alpha^2 \sum_{m=1}^N \left(\sum_{n=1}^N \int k \frac{\chi_\Lambda(k)}{\omega(k)} \int_u^T \int_0^t [1 + \Theta(s-u)] e^{-\omega(k)(t-s)} e^{-k^2(t-u)/2} e^{-k^2(s-u)_+/2} \right. \\ & \quad \left. \times \sin[k(X_{s \wedge u}^n - X_u^m)] ds dt dk \right)^2 \\ & \equiv \alpha^2 \sum_{m=1}^N \left(\sum_{n=1}^N \mathcal{C}_{m,n} \right)^2. \end{aligned} \quad (4.12)$$

We will now bound the absolute value of each one of the terms $\mathcal{C}_{m,n}$ from above. We first note that all angular integrals involved may be performed explicitly, obtaining the following expression for $\mathcal{C}_{m,n}$,

$$4\pi \int_u^T \int_0^t \frac{X_{s \wedge u}^n - X_u^m}{|X_{s \wedge u}^n - X_u^m|} [1 + \Theta(s-u)] \int_0^\Lambda \frac{e^{-\nu(r)(t-s)}}{\nu(r)} e^{-r^2(t-u)/2} e^{-r^2(s-u)_+/2} \varphi(r|X_{s \wedge u}^n - X_u^m|) r^3 dr ds dt, \quad (4.13)$$

where $\nu(r) \equiv \sqrt{r^2 + \mu^2}$ and $\varphi(x) \equiv (\sin x - x \cos x)/x^2$. We then find that, by bounding from above by the massless case $\nu(r) \geq r$, and splitting the s -integral into the fragments corresponding to $[0, u]$ and $[u, t]$,

$$\begin{aligned} |\mathcal{C}_{m,n}| & \leq 4\pi \int_0^\Lambda \int_0^u \frac{1 - e^{-(r+r^2/2)(T-u)}}{1 + r/2} r e^{-r(u-s)} |\varphi(r|X_u^m - X_s^n|)| ds dr \\ & \quad + 8\pi \int_0^\Lambda \int_u^T \int_u^t e^{-r(t-s)} e^{-r^2(t-u)/2} e^{-r^2(s-u)/2} |\varphi(r|X_u^m - X_s^n|)| r^2 ds dt dr \\ & \equiv \mathcal{D}_{m,n} + \mathcal{E}_{m,n}. \end{aligned} \quad (4.14)$$

We will now estimate \mathcal{D} and \mathcal{E} in two separate lemmas.

Lemma 4.1. *For all $\varepsilon > 0$, $0 \leq \phi < 1$, and $1 < \theta < 2$,*

$$\mathcal{D}_{m,n} \leq \frac{2^{2+\phi} \pi \|\varphi\|_\infty \Gamma(2-\phi)}{(1-\phi) \varepsilon^{1-\phi}} + 2\sqrt{2} \pi \|\varphi(x) x^{\theta/2}\|_\infty \int_0^\infty \frac{r^{1/2-\theta/2}}{1+r/2} dr \left(\int_0^u \frac{1_{[0,\varepsilon]}(u-s)}{|X_u^m - X_s^n|^\theta} ds \right)^{1/2}. \quad (4.15)$$

Proof. Let $\varepsilon > 0$. We have that

$$\begin{aligned} \mathcal{D}_{m,n} & \leq 4\pi \int_0^u \int_0^\infty \frac{r e^{-r(u-s)}}{1+r/2} |\varphi(r|X_u^m - X_s^n|)| dr ds \\ & = 4\pi \int_0^{(u-\varepsilon)_+} \int_0^\infty \frac{r e^{-r(u-s)}}{1+r/2} |\varphi(r|X_u^m - X_s^n|)| dr ds \\ & \quad + 4\pi \int_{(u-\varepsilon)_+}^u \int_0^\infty \frac{r e^{-r(u-s)}}{1+r/2} |\varphi(r|X_u^m - X_s^n|)| dr ds. \end{aligned} \quad (4.16)$$

We will first concentrate on the first term. First we notice $1/(1+r/2)$ is both bounded above by 1 and $2/r$, which implies that it is bounded by $2^\phi/r^\phi$ for all $0 \leq \phi \leq 1$, by interpolation. By using this result, we get that, for all $0 \leq \phi < 1$,

$$\begin{aligned}
& 4\pi \int_0^{(u-\varepsilon)_+} \int_0^\infty \frac{re^{-r(u-s)}}{1+r/2} |\varphi(r|X_u^m - X_s^n)| dr ds \leq 2^{2+\phi} \pi \|\varphi\|_\infty \int_0^{(u-\varepsilon)_+} \int_0^\infty r^{1-\phi} e^{-r(u-s)} dr ds \\
& = 2^{2+\phi} \pi \|\varphi\|_\infty \int_0^\infty r^{1-\phi} e^{-r} dr \int_0^{(u-\varepsilon)_+} (u-s)^{\phi-2} ds \\
& = \frac{2^{2+\phi} \pi \|\varphi\|_\infty \Gamma(2-\phi)}{1-\phi} \left\{ \frac{1}{[u-(u-\varepsilon)_+]^{1-\phi}} - \frac{1}{u^{1-\phi}} \right\} \leq \frac{2^{2+\phi} \pi \|\varphi\|_\infty \Gamma(2-\phi)}{(1-\phi)\varepsilon^{1-\phi}}.
\end{aligned} \tag{4.17}$$

As for the second term, we first note that the small and large x -behavior of $\sin x - x \cos x$ show that $\varphi(x)x^a$ is bounded for all $a \in [-1, 1]$ and that, moreover, for no other values of a is $\varphi(x)x^a$ bounded. Therefore, one can bound $|\varphi(x)|$ as $\|\varphi(x)x^a\|_\infty |x|^{-a}$, and by doing this we find

$$\begin{aligned}
& 4\pi \int_{(u-\varepsilon)_+}^u \int_0^\infty \frac{re^{-r(u-s)}}{1+r/2} |\varphi(r|X_u^m - X_s^n)| dr ds \\
& \leq 4\pi \|\varphi(x)x^a\|_\infty \int_0^\infty \frac{r^{1-a}}{1+r/2} \int_{(u-\varepsilon)_+}^u e^{-r(u-s)} |X_u^m - X_s^n|^{-a} ds dr \\
& \leq 4\pi \|\varphi(x)x^a\|_\infty \int_0^\infty \frac{r^{1-a}}{1+r/2} \left(\int_{(u-\varepsilon)_+}^u e^{-2r(u-s)} ds \right)^{1/2} \left(\int_{(u-\varepsilon)_+}^u |X_u^m - X_s^n|^{-2a} ds \right)^{1/2} dr \\
& \leq 2\sqrt{2}\pi \|\varphi(x)x^a\|_\infty \int_0^\infty \frac{r^{1/2-a}}{1+r/2} dr \left(\int_{(u-\varepsilon)_+}^u |X_u^m - X_s^n|^{-2a} ds \right)^{1/2}.
\end{aligned} \tag{4.18}$$

□

Lemma 4.2. For $m \neq n$,

$$\mathcal{E}_{m,n} \leq 4\pi \|\varphi(x)/x\|_1. \tag{4.19}$$

Proof. By applying Fubini's Theorem we find that

$$\begin{aligned}
\mathcal{E}_{m,n} &= 8\pi \int_0^\Lambda \int_u^T \int_s^T e^{-r(t-s)} e^{-r^2(t-u)/2} e^{-r^2(s-u)/2} |\varphi(r|X_u^m - X_u^n)| r^2 dt ds dr \\
&= 8\pi \int_0^\Lambda \int_u^T \int_s^T e^{-(r+r^2/2)(t-s)} e^{-r^2(s-u)} |\varphi(r|X_u^m - X_u^n)| r^2 dt ds dr \\
&\leq 8\pi \int_0^\infty \int_u^\infty \int_s^\infty e^{-(r+r^2/2)(t-s)} e^{-r^2(s-u)} |\varphi(r|X_u^m - X_u^n)| r^2 dt ds dr \\
&= 8\pi \int_0^\infty \frac{|\varphi(r|X_u^m - X_u^n)|}{r(1+r/2)} dr \leq 8\pi \int_0^\infty \frac{|\varphi(r|X_u^m - X_u^n)|}{r} dr = 4\pi \|\varphi(x)/x\|_1.
\end{aligned} \tag{4.20}$$

□

We now conclude from the previous two lemmas that

$$\begin{aligned}
|\mathcal{C}_{m,n}| &\leq 4\pi \|\varphi(x)/x\|_1 + \frac{2^{2+\phi} \pi \|\varphi\|_\infty \Gamma(2-\phi)}{(1-\phi)\varepsilon^{1-\phi}} \\
&\quad + 2\sqrt{2}\pi \|\varphi(x)x^{\theta/2}\|_\infty \int_0^\infty \frac{r^{1/2-\theta/2}}{1+r/2} dr \left(\int_0^u \frac{1_{[0,\varepsilon]}(u-s)}{|X_u^m - X_s^n|^\theta} ds \right)^{1/2} \\
&\equiv C + D_\varepsilon + F_\theta \left(\int_0^u \frac{1_{[0,\varepsilon]}(u-s)}{|X_u^m - X_s^n|^\theta} ds \right)^{1/2},
\end{aligned} \tag{4.21}$$

with the first term removed when $m = n$. It follows then that, by means of the inequality $(a+b)^2 \leq 2(a^2+b^2)$,

$$\begin{aligned} E(D_u \mathcal{A}_T | \mathcal{F}_u)^2 &= \alpha^2 \sum_{m=1}^N \left(\sum_{n=1}^N \mathcal{C}_{m,n} \right)^2 \leq \alpha^2 N \sum_{m=1}^N \sum_{n=1}^N \mathcal{C}_{m,n}^2 \\ &\leq 2\alpha^2 N^2 (N-1)(C + D_\varepsilon)^2 + 2\alpha^2 N^2 D_\varepsilon^2 + 2\alpha^2 N F_\theta^2 \sum_{m,n} \int_0^u \frac{1_{[0,\varepsilon]}(u-s)}{|X_u^m - X_s^n|^\theta} ds. \end{aligned} \quad (4.22)$$

4.3 The lower bound

By using the estimates (1.7), (4.22) we obtain

$$\begin{aligned} E(e^{\mathcal{A}_T}) &\leq e^{E(\mathcal{A}_T)} E \left[\exp \left(\frac{p^2}{2(p-1)} \int_0^T E(D_t \mathcal{A}_T | \mathcal{F}_t)^2 dt \right) \right]^{1-1/p} \\ &\leq e^{E(\mathcal{A}_T)} e^{\gamma T} E \left[\exp \left(\sum_{m,n} \beta \int_0^T \int_0^t \frac{1_{[0,\varepsilon]}(t-s)}{|X_t^m - X_s^n|^\theta} ds dt \right) \right]^{1-1/p}, \end{aligned} \quad (4.23)$$

where

$$\begin{aligned} \gamma &\equiv p\alpha^2 N^2 (N-1)(C + D_\varepsilon)^2 + p\alpha^2 N^2 D_\varepsilon^2 = p\alpha^2 N^2 (N-1)(C^2 + 2CD_\varepsilon) + p\alpha^2 N^3 D_\varepsilon^2, \\ \beta &\equiv \frac{\alpha^2 p^2 N F_\theta^2}{p-1}, \end{aligned} \quad (4.24)$$

and, appealing to the combinatorial argument presented in Sections 2 and 3, we get

$$\begin{aligned} &E \left[\exp \left(\beta \sum_{m,n} \int_0^T \int_0^t \frac{1_{[0,\varepsilon]}(t-s)}{|X_t^m - X_s^n|^\theta} ds dt \right) \right] \\ &\leq E \left[\exp \left(N\beta \int_0^T \int_0^t \frac{1_{[0,\varepsilon]}(t-s)}{|Y_t - Y_s|^\theta} ds dt \right) \right] E \left[\exp \left(2N\beta \int_0^T \int_0^t \frac{1_{[0,\varepsilon]}(t-s)}{|Y_t - Z_s|^\theta} ds dt \right) \right]^{(N-1)/2}. \end{aligned} \quad (4.25)$$

Similarly to what we did for the polaron model above, we will now utilize the following particular forms of the bounds found in Theorems 2.2 and 2.3 from the work by the author and Lawrence Thomas [3],

$$\begin{aligned} E \left[\exp \left(\lambda \int_0^T \int_0^t \frac{1_{[0,\varepsilon]}(t-s)}{|Y_t - Y_s|^\theta} ds dt \right) \right] &\leq \exp \left[\left(A_\theta \lambda^{2/(2-\theta)} \varepsilon^{2/(2-\theta)} + \frac{B_\theta \varepsilon^{1-\theta/2} \lambda}{1-\theta/2} \right) T \right], \\ E \left[\exp \left(\lambda \int_0^T \int_0^t \frac{1_{[0,\varepsilon]}(t-s)}{|Y_t - Z_s|^\theta} ds dt \right) \right] &\leq \exp \left[2^{-\theta/(2-\theta)} A_\theta \lambda^{2/(2-\theta)} \varepsilon^{2/(2-\theta)} T + o(T) \right], \end{aligned} \quad (4.26)$$

where A_θ and B_θ are explicit functions of θ ,

$$A_\theta = \frac{2^{(3\theta-2)/(2-\theta)} \theta^{\theta/(2-\theta)} (2-\theta)}{(3-\theta)^{2\theta/(2-\theta)}}, \quad B_\theta = \frac{\theta \Gamma[(3-\theta)/2]}{2^{\theta/2} \Gamma(3/2)}. \quad (4.27)$$

We then find that

$$\begin{aligned} E(e^{\mathcal{A}_T}) &\leq e^{E(\mathcal{A}_T)} e^{\gamma T} \exp \left[\left(A_\theta N^{2/(2-\theta)} \beta^{2/(2-\theta)} \varepsilon^{2/(2-\theta)} + \frac{B_\theta \varepsilon^{1-\theta/2} N \beta}{1-\theta/2} \right) \frac{(p-1)T}{p} \right] \\ &\quad \times \exp \left[A_\theta N^{2/(2-\theta)} (N-1) \beta^{2/(2-\theta)} \varepsilon^{2/(2-\theta)} (p-1) p^{-1} T + o(T) \right] \\ &= e^{E(\mathcal{A}_T)} e^{\gamma T} \exp \left[\left(A_\theta N^{(4-\theta)/(2-\theta)} \beta^{2/(2-\theta)} \varepsilon^{2/(2-\theta)} + \frac{B_\theta \varepsilon^{1-\theta/2} N \beta}{1-\theta/2} \right) \frac{(p-1)T}{p} + o(T) \right], \end{aligned} \quad (4.28)$$

and this is how we finally conclude that, from the Feynman-Kac formula (1.5),

$$\begin{aligned} E_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda} &\geq -p\alpha^2 N^2(N-1)(C^2 + 2CD_\varepsilon) - p\alpha^2 N^3 D_\varepsilon^2 \\ &\quad - A_\theta N^{(6-\theta)/(2-\theta)} \alpha^{4/(2-\theta)} F_\theta^{4/(2-\theta)} p^{(2+\theta)/(2-\theta)} (p-1)^{-\theta/(2-\theta)} \varepsilon^{2/(2-\theta)} \\ &\quad - \alpha^2 B_\theta \varepsilon^{1-\theta/2} N^2 p F_\theta^2 (1-\theta/2)^{-1}. \end{aligned} \quad (4.29)$$

There are several parameters involved, and fully optimizing the expression above in all of them is beyond the point we want to make here, that using a simple argument involving the Clark-Ocone formula one can find an explicit lower bound for $E + Q$. Many particular such bounds may be derived from expression (4.29). For example, we may first pick $\theta = 3/2$ and then choose $\varepsilon = N^{-2}\alpha^{-2}$. The lower bound then becomes

$$E_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda} \geq -p\alpha^2 N^2(N-1)(C^2 + 2CD_\varepsilon) - p\alpha^2 N^3 D_\varepsilon^2 - LN - M\alpha^{3/2} N^{3/2}, \quad (4.30)$$

where L and M are constants. Even though the N and α behavior of the last two terms is clear, for the first two there is still a degree of freedom given by ε . In particular, the second term may be bounded above as

$$U\alpha^2 N^3 \frac{(N^4 \alpha^4)^{1-\phi}}{(1-\phi)^2}, \quad (4.31)$$

where U is a constant. Assuming now that $N^2 \alpha^2 \geq e$ we may select $1-\phi = 1/\log(N^2 \alpha^2)$. The expression then becomes

$$4Ue^2 \alpha^2 N^3 \log^2(\alpha N), \quad (4.32)$$

where we used the fact that $x^{1/\log x}$ is e for all $x > 0$. A similar analysis may be applied to the first term, leading to a bound of a similar form, except that the logarithmic term is now raised to the first power. After all these computations we conclude that

$$E_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda} \geq -D\alpha^2 N^3 \log^2(\alpha N), \quad (4.33)$$

for large enough α , where D is an explicit positive constant. Now, if α is sufficiently small, it is easy to deduce a lower bound of the form $-D\alpha^2 N^3 \log^2 N$ when $N \geq 2$, and $-D\alpha^2$ when $N = 1$, by performing a similar procedure (one can choose $\varepsilon = N^{-2}$ when $N \geq 2$, for instance).

One would like now to have an upper bound for $E + Q$. For the moment, all we can provide is a rather trivial upper bound, which follows by Jensen's inequality: Recall the functional integral that determines the ground-state energy of the Nelson model (see equation (4.1)),

$$E \left[\exp \left(\alpha \sum_{m,n} \int_0^T \int_0^t \int \frac{e^{-ik(X_t^m - X_s^n)}}{\omega(k)} e^{-\omega(k)(t-s)} \chi_\Lambda(k) dk ds dt \right) \right]; \quad (4.34)$$

Jensen's inequality then allows us to bound this from above as

$$\exp \left\{ E \left[\alpha \sum_{m,n} \int_0^T \int_0^t \int \frac{e^{-ik(X_t^m - X_s^n)}}{\omega(k)} e^{-\omega(k)(t-s)} \chi_\Lambda(k) dk ds dt \right] \right\} = \exp [Q_{\alpha,\mu}^{N,\Lambda} T + o(T)] \quad (4.35)$$

(the equality following from the discussion in Subsection 4.1), which implies, by the Feynman-Kac formula (1.5)

$$E_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda} \leq 0. \quad (4.36)$$

References

- [1] R.D. Benguria, G.A. Bley, Exact asymptotic behavior of the Pekar-Tomasevich functional, J. Math. Phys. 52, 052110 (2011).

- [2] G.A. Bley; Estimates on Functional Integrals of Non-Relativistic Quantum Field Theory, with Applications to the Nelson and Polaron Models; Ph.D. thesis; University of Virginia Library (2016).
- [3] G.A. Bley, L.E. Thomas, Estimates on Functional Integrals of Quantum Mechanics and Non-Relativistic Quantum Field Theory, arXiv:1512.00356 (2015).
- [4] C.J. Coulborn, J.H. Dinitz, Handbook of Combinatorial Designs, 2nd edition, Chapman & Hall/CRC (2006).
- [5] R.P. Feynman, Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction, Phys. Rev. 80, 440 (1950).
- [6] R.P. Feynman, Slow Electrons in a Polar Crystal, Phys. Rev., 97, 3 (1955).
- [7] H. Fröhlich, Electrons in lattice fields, Adv. Phys. 3, 325 (1954).
- [8] M. Griesemer, J.S. Møller, Bounds on the Minimal Energy of Translation Invariant N -Polaron Systems, Comm. Math. Phys. 297, 283-297 (2010).
- [9] M. Gubinelli, F. Hiroshima, J. Lörinczi, Ultraviolet renormalization of the Nelson Hamiltonian through functional integration, J. Funct. Anal., 267, 3125-3153 (2014).
- [10] I. Karatzas, S. Shreve, Brownian Motion and Stochastic Calculus, 2nd ed., Springer, New York, (1991).
- [11] I. Karatzas, S. Shreve, Methods of Mathematical Finance, Springer, (2001).
- [12] E.H. Lieb, M. Loss, Analysis, 2nd ed., AMS (2011).
- [13] E.H. Lieb, K. Yamazaki, Ground-state energy and effective mass of the polaron, Phys. Rev. 111, 728-733 (1958).
- [14] J. Lörinczi, R.A. Minlos, H. Spohn, The Infrared Behaviour in Nelson's Model of a Quantum Particle Coupled to a Massless Scalar Field, Ann. Henri Poincaré, 269-295 (2002).
- [15] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, J. Math. Phys. 5, 1190-1197 (1964).
- [16] E. Nelson, Schrödinger particles interacting with a quantized scalar field, in Analysis in Function Space: Proceedings of a conference on the theory and application of analysis in function space held at Endicott House in Dedham, Mass., June 9-13, 1963, W.T. Martin and I. Segal, eds., 87-121, MIT Press, Cambridge, Mass., (1964).
- [17] D. Nualart, The Malliavin Calculus and Related Topics, 2nd. ed., Springer, Berlin (2006).
- [18] S.I. Pekar, Untersuchung über die Elektronentheorie der Kristalle, Akademie Verlag, Berlin (1954).